Scientific report on a visit to the University of Warwick

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I visited the University of Warwick during 9-14.9.2012 with the support of a grant within the ESF activity entitled 'Applied and Computational Algebraic Topology' (ACAT). The purpose of the visit was to initiate a research project with Prof. Michael Farber on various aspects of the theory of random simplicial complexes.

Our work during this week was mostly concerned with random hypertrees. A k-hypertree is a k-dimensional simplicial subcomplex K of the $(n-1)$-simplex Δ_{n-1} such that K contains the full $(k-1)$-dimensional skeleton of Δ_{n-1} and such that K is $\mathbb{Q}$-acyclic, i.e. $\tilde{H}_*(K;\mathbb{Q}) = 0$. As the higher dimensional counterparts of the usual 1-dimensional graphical trees, these hypertrees are expected to play an important role in the study of general simplicial complexes. While there is a highly developed theory of random spanning trees of a graph, relatively little is known in the higher dimensional case. During the visit Prof. Farber, Dr. Armindo Costa (a postdoc of Prof. Farber) and I concentrated on a number of aspects of random hypertrees:

1) A central result on k-hypertrees is Kalai's extension of Cayley's tree enumeration formula. Let $\mathcal{C}(n,k)$ denote the family of all k-hypertrees in Δ_{n-1}. Kalai proved that

$$\sum_{K \in \mathcal{C}(n,k)} |\tilde{H}_{k-1}(K)|^2 = n^{\binom{n-2}{k}}. \tag{1}$$

We found a simplification of Kalai's original approach which leads to a short (and essentially computation free) proof of (1).

2) Many results on random trees are obtained by analyzing stochastic processes that generate a random tree with a prescribed (say uniform) distribution. While several such processes are known in the 1-dimensional case (e.g. the famous "groundskeeper" algorithm), the situation is more involved in higher dimensions. We defined and studied a certain Markov process that *conjecturally* generates uniform random k-hypertrees of Δ_{n-1}. I've written computer simulations that suggest that the output distribution is indeed uniform and further indicate that $\tilde{H}_{k-1}(K)$ is typically "nearly" cyclic. Our main result at this point is a precise description of the local changes in the cardinality $|\tilde{H}_{k-1}(K)|$ of the $(k-1)$-th homology of K during this process.

Prof. Farber and I plan to continue our collaboration on this project and to extend the preliminary results we obtained during this short visit. We hope our joint work will eventually result in a paper on random hypertrees.