FINAL REPORT - ESF SHORT VISIT GRANT

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The main purpose of my stay in the United States in October and November 2012 was to work on an ongoing joint research project with Prof. Marek Biskup (UCLA) and Michele Salvi (TU Berlin). In this project, we consider the *Dirichlet energy* that represents the total current flow in an electrostatic resistor network if the potential is kept at a fixed voltage in a subset of the network. The particular network we consider is the so-called *random conductance model (RCM)*, where every edge in the lattice $\mathbb{Z}^d$ is assigned a random positive weight standing for the conductance between adjacent vertices. The main goal of the project is to prove a central limit theorem (CLT) for the effective conductance over a bounded set with prescribed boundary conditions if the volume of the set is large.

1. Introduction

Consider the lattice $\mathbb{Z}^d$ and assign to any edge (x, y) connecting two neighboring sites $x \sim y$ a random weight $\omega_{x,y} \in [0, \infty)$, subject to the symmetry condition $\omega_{x,y} = \omega_{y,x}$. The quantity $\omega_{x,y}$ is referred to as *conductance*, whereas its reciprocal is called *resistance*. Denote by $\mathcal{N} = \{e_1, \dots, e_d\}$ the canonical base of $\mathbb{Z}^d$, i.e., the set of neighbors of the origin with nonnegative entries. We assume that $(\omega_{x,x+e})_{x\in\mathbb{Z}^d, e\in\mathcal{N}}$ is a family of positive i.i.d. random variables, and we abbreviate $\omega(x, e) = \omega_{x,x+e}$. Define the randomly perturbed Laplacian Δ^ω by

$$\Delta^\omega f(x) := \sum_{y\in\mathbb{Z}^d :\, y\sim x} \omega_{x,y}(f(y) - f(x)). \tag{1.1}$$

The large-scale behavior of this operator has been a central object of study for a long time. A very common problem is the analysis of solutions to associated Poisson equations on large domains, where the conductances are assumed to be periodic or ergodic. These are the most prominent problems in discrete *homogenisation theory*. We will have a look at a related problem, namely the one of the Dirichlet energy. For that purpose, let us assume that the conductances $(\omega_{x,y})_{x\sim y\in\mathbb{Z}^d}$ are distributed according to some probability measure $\mathbb{P}$ satisfying

i) the $\omega_{x,y}$ with $x \sim y \in \mathbb{Z}^d$ under $\mathbb{P}$ are i.i.d.
ii) there exists $\lambda \in (0, 1)$ such that $\lambda \le \omega_{x,y} \le \frac{1}{\lambda}$ for all $x \sim y \in \mathbb{Z}^d$ almost surely w.r.t. $\mathbb{P}$ (uniform ellipticity).

In particular, the conductances constitute a stationary ergodic environment under these assumptions. Now consider the box $\Lambda_L = [0, L] \cap \mathbb{Z}^d$. For a function $f : \Lambda_L \cup \partial\Lambda_L \to \mathbb{R}$, the Dirichlet energy associated with the potential f is defined as

$$Q_L(f) = \sum_{x\sim y,\, \{x,y\}\cap\Lambda_L\neq\emptyset} \omega_{x,y}\big(f(y) - f(x)\big)^2.$$

We are interested in the so-called *effective conductance*

$$Q_L^t = \inf\Big\{Q_L(f) :\ f(z) = t\cdot z \quad \forall z \in \partial\Lambda_L\Big\}$$

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with $t \in \mathbb{R}^d$. As is known from electrostatic theory,

$$Q_L^t = Q_L(t \cdot \psi_L)$$

where $\psi_L = (\psi_L^1, \ldots, \psi_L^d)$ and $\psi_L^i \colon \Lambda_L \to \mathbb{R}$ for $i = 1 \ldots, d$ is the unique function that satisfies

$$\begin{aligned} \Delta^\omega \psi_L^i &= 0, \qquad \text{on } \Lambda_L \\ \psi_L^i(z) &= z_i, \qquad z \in \partial\Lambda_L. \end{aligned}$$

At least under the assumptions on the conductance distribution, it can also be shown that there also exists a corresponding infinite-volume object, the so called *harmonic coordinate* $\psi \colon \mathbb{Z}^d \to \mathbb{R}^d$ all d components of which are harmonic everywhere on the lattice, that is, $\Delta^\omega \psi^i(z) = 0$ for $z \in \mathbb{Z}^d$. Moreover, we may choose ψ such that $\psi(0) = 0$. For large L, it is known that

$$Q_L^t \sim Q_L(t \cdot \psi) \sim |t|^2 L^d a$$

where

$$a = \mathbb{E}\big[\sum_{x \sim 0} \omega_{0,x} |\psi(x)|^2\big]$$

with $\mathbb{E}$ standing for the $\mathbb{P}$-expectiation. The quantity a is referred to as *effective conductivity constant* which accounts for the fact that on large scales Δ^ω behaves in many ways like the continuous Laplacian multiplied with this quantity, compare e.g. [B11]. When considering discrete differential equations on the lattice w.r.t. the elliptic operator Δ^ω, it is therefore of considerable importance to control the fluctuations of Q_L^t.

2. Current state of the research project

We have established a framework to prove a central limit theorem for the effective conductance Q_L^t which seems likely to admit an extension to a broader class of boundary conditions (e.g., mixed Dirichlet-Neumann) and domains. The first step is to represent the deviation of Q_L^t from its mean by the Doob martingale we obtain by conditioning on the edges contributing to Q_L^t. Here choosing a stationary ordering of these edges is of vital importance. Each martingale increment admits a representation in terms of the gradient of $t \cdot \psi_L$. In order to employ the martingale CLT, it is essential to rely on the ergodicity of the environment and replace ψ_L by the shift-covariant full-lattice analogue ψ. Effectively, this requires controlling the expectation of

$$L^{-d} \sum_e |t \cdot \nabla_e(\psi_L - \psi)|^4 = L^{-d} \sum_e |t \cdot \nabla_e(\chi_L - \chi)|^4, \tag{2.1}$$

here the sum includes all edges in the box Λ_L and the so called *correctors* χ_L resp. χ are given by

$$\chi_L(z) = \psi_L(z) - z, \qquad \chi(z) = \psi(z) - z.$$

A uniform estimate of the form

$$\mathbb{E}\big[\|\nabla \chi_L\|_{\ell^p(\Lambda_L)}\big] \lesssim L^d \tag{2.2}$$

for some $p > 4$ is needed to gain control over the expectation of (2.1) for $L \to \infty$ by uniform p-integrability. However, we have been able to show this only under a restriction to sufficiently small ellipticity contrast, that is, for λ sufficiently close to 1. We have presented our result in the article *A central limit theorem for the effective conductance: I. Linear boundary data and small ellipticity contrasts* which was published on the arXiv (`http://arxiv.org/abs/1210.2371`) recently. The purpose of the visit to Los Angeles was to find a strategy to drop this additional requirement.

3. Progress made during the stay

We were able to identify two different approaches towards establishing a uniform L^p-estimate in the spirit of (2.2) that holds without the requirement of a sufficiently small ellipticity contrast.

The first one is limited to the case $d \geq 3$ and uses one of the main results from the recent paper [GO11]. The authors show that a stationary version of the corrector χ (the statinary version obviously does not have the property that it vanishes in zero) satisfies

$$\mathbb{E}[|\chi(0)|^p] < \infty \qquad \text{for all } p \geq 2. \tag{3.1}$$

In addition, it is obvious that $t \cdot (\psi_L^i - \psi^i)$ is the unique real function that is harmonic on Λ and agrees with $t \cdot \chi$ on the boundary $\partial\Lambda$. We have reason to believe that the regularity of χ on the boundary extends to the full-box object by certain harmonicity properties.

The second strategy is a perturbative argument. It can be shown that the second derivative of the finite-volume harmonic coordinate with regard to the value of a single conductance takes a purely multiplicative form:

$$\frac{\partial^2}{\partial^2\omega(z,e_j)}\psi_L^i(x) = \nabla_{e_i}^{(1)}\nabla_{e_j}^{(2)}G_L^\omega(z,z)\frac{\partial}{\partial\omega(z,e_j)}\psi_L^i(x) \tag{3.2}$$

with G_L^ω the corresponding finite-volume Green's function. The double gradient of the Green's function in a diagonal point is quite well understood and monotonous in the conductances. By (3.2), we are able to relate the impact that a large single-edge perturbation of the conductances has on the harmonic coordinate by the corresponding effect of a small perturbation, independently of the vertex in consideration. More precisely, if η is a conductance configuration that is zero except at the edge $(z, z + e_j)$, where it is one, and a, b are positive numbers, we obtain the formula

$$\psi_L^i(\omega + b\eta, x) - \psi_L^i(\omega, x) = [\psi_L^i(\omega + a\eta, x) - \psi_L^i(\omega, x)]\frac{C_{z,j}(b)}{C_{z,j}(a)}. \tag{3.3}$$

Here, the positive numbers $C_{z,j}$ are given by

$$C_{z,j}(a) = \int_0^a \exp\Big\{\int_0^s \nabla_{e_i}^{(1)}\nabla_{e_j}^{(2)}G_L^{\omega+t\eta}(z,z)\,\mathrm{d}t\Big\}\,\mathrm{d}s \tag{3.4}$$

and, in particular, are independent of x and i, thus (3.3) holds analogously for norms of ψ_L^i. Choosing b large and a so small that $\omega + a\eta$ is still within the range of small ellipticity contrast suggests that regularity of the involved objects extends to broader ellipticity contrasts by iterative arguments.

4. Future collaboration and projected publication

Close collaboration between Marek Biskup, Michele Salvi and myself is going on on a continuous basis, an additional meeting of the three collaborators in Europe is planned in the near future.

Finally, let me express my gratitude for the support the ESF provided for me in order to realize this research project, which has been and continues to be of great importance for my career and education.

Berlin, November 2012 - Tilman Wolff

References

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