Scientific Report

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March 14, 2013

1 Scientific Report concerning the ESF-Short Visit Grant with Reference Number 5398

In the framework of the ESF activity entitled 'Contact and Symplectic Topology' I was awarded a grant amounting to a maximum of 1010 EUR, which includes travel costs up to a maximum of 500 EUR, to spend 6 days at Tel Aviv University in Israel hosted by Professor Yaron Ostrover. This grant has reference number 5398.

1.1 Purpose of the visit

Professor Yaron Ostrover and I, we decided to work on the relationship between the notion of a symplectic capacity and mathematical billards. In particular, there is a capacity called the Hofer-Zehnder capacity which represents the length of the shortest periodic (K, T)-billiard trajectory measured with respect to the support function of T. $K, T \subset \mathbb{R}^n$ are two convex bodies with smooth boundary in n-dimensional real space.

It was shown by Prof. Ostrover that there exists a so called dichotomy between proper and gliding (K, T)-billiard trajectories, meaning that any closed (K, T)-billiard trajectory is either a proper or a gliding one.

The puropose of this visit was to investigate and to understand whether the following outstanding question can be answered to be true or false, for more details, please see [1].

Problem 1.1 *These closed (K, T)-billiard trajectories are critical points of a certain functional, essentially an energy functional on the space of one time weakly differentiable and square-integrable periodic mappings into $\Sigma = K \times T$,*

$$I_\Sigma(z) = \int_0^{2\pi} h_\Sigma^2(\dot{z}(t))\, dt,$$

where Σ is a subset of $2n$-dimensional real space. The question is whether it can be shown that such a gliding (K, T)-billiard trajectory can not appear as a local minimum of this functional I_Σ.

In two dimensions the problem is solved, but in higher dimensions it remains open.

1.2 Description of the work carried out during the visit

In January 2013 I met Prof. Ostrover on the occasion of a Workshop on "Symplectic Geometry, Contact Geometry and Interactions" in Les Diablerets, Switzerland. During this conference we talked about the above mentioned question, whether a gliding billiard trajectory can be a (local) minimal of this functional. In the subsequent weeks, in advance of my visit in Tel Aviv, I worked through research articles of Prof. Ostrover, I read parts of books on mathematical billards and finally, I learnt a lot of convex geometry and advanced methods of functional analysis.

In Tel Aviv, I had a working place at the Mathematical Institute. Prof. Ostrover had his office in the same building, so without big efforts it was possible to discuss ideas and to share thoughts. I also attended a couple of talks.

1.3 Description of the main results obtained

First, I went through a couple of steps by Prof. Ostrover towards a proof of the problem. And I confirmed, that these steps are not sufficient to decide it. Inbetween these steps, there is the following lemma.

Lemma 1.2 (Artstein-Avidan, Ostrover (private communication)) *Let $\Sigma \subset \mathbb{R}^{2n}$ be a convex body. If $z \in \mathcal{E}$ is a local minimum of $I_\Sigma(z) = \int_0^{2\pi} h_\Sigma^2(\dot{z}(t))\, dt$ such that h_Σ is twice differentiable at all points $\dot{z}(t)$, then for any $\xi \in W^{1,2}(S^1, \mathbb{R}^{2n})$ with $\int_0^{2\pi} \xi(t)\, dt = 0$ one has:*

$$\int_0^{2\pi} \left\langle \left(Hess\left(h_\Sigma^2\right)(\dot{z}(t))\right) \dot{\xi}(t) \mid \dot{\xi}(t) \right\rangle dt \geq I_\Sigma(z) \int_0^{2\pi} \left\langle J\xi(t) \mid \dot{\xi}(t) \right\rangle dt.$$

I used this Lemma to prove that if for a critical point z of the functinal I_Σ it holds that the biggest eigenvalue of the hessian of $h_\Sigma^2(\dot{z}(t))$ is (strictly) smaller than the value $I_\Sigma(z)$, then it follows, that z is not a (local) minimum of I_Σ, more precisely

Lemma 1.3 *Let $\Sigma \subset \mathbb{R}^{2n}$ be a convex body. If $z \in \mathcal{E}$ is a critical point of I_Σ satisfying*

$$\left\| Hess\left(h_\Sigma^2\right)(\dot{z}(t)) \right\| < I_\Sigma(z),$$

then it follows that z can not be a local minimum of I_Σ. Moreover, since we know that a gliding (K,T)-billiard trajectory γ must be of the form

$$\dot{\gamma}(t) = \begin{pmatrix} \dot{\gamma}_q(t) \\ \dot{\gamma}_p(t) \end{pmatrix} = \begin{pmatrix} -\alpha(t)\nabla g_T(\gamma_p(t)) \\ \beta(t)\nabla g_K(\gamma_q(t)) \end{pmatrix},$$

where α and β are smooth and positive functions, we have for $z := \mathcal{F}(\gamma)$ that the condition

$$\left\| Hess\left(h_\Sigma^2\right)(\dot{z}(t)) \right\| < \alpha(t) + \beta(t)$$

implies in the same way that z can not serve as a local minimum of I_Σ.

The application $\mathcal{F}$ is a correspondence between generalized closed characteristics on $\partial\Sigma$ and (weak) critical points of the functional I_Σ, for further details, please consult [1]. This is a new approach to the problem. Lemma 1.3 suggests that one can start to try to calculate the two functions α and β for special classes of Σ to get first insights. For example, one knows that if α (β) is a constant function, hence it is β (α).

I have deduced Lemma 1.3 by introducing the following differential equation:

$$\operatorname{Hess}(h_\Sigma^2)(\dot{z}(t))\dot{\xi}(t) - I_\Sigma(z)J\xi(t) = -M\dot{\xi}(t),$$

where M is an appropriate positive definite matrix. If one chooses $M := I_\Sigma(z)\mathrm{Id} - \operatorname{Hess}(h_\Sigma^2)(\dot{z}(t))$, then one can deduce the statement above by solving this differential equation:

$$\xi(t) = \exp(tJ)\xi_0, \quad \xi_0 \in \mathbb{R}^{2n}.$$

And this solution indeed satisfies the condition $\int_0^{2\pi} \xi(t)\, dt = 0$ and it can even be normalized (by adjusting the constants correctly), meaning that the action of ξ equals one:

$$\frac{1}{2}\int_0^{2\pi} \left\langle J\xi(t) \mid \dot{\xi}(t) \right\rangle dt = 1,$$

what implies that $\xi \in \mathcal{E}$. This method can be used to deduce other statements on (local) minima of the functional I_Σ.

1.4 Future collaboration with host institution

During my stay in Tel Aviv, in particular as a result of our discussions, new ideas and insights concerning this problem came up. For example different ideas on how to find a gliding (K, T)-billiard trajectory (as a critical point of I_Σ) contradicting the inequality given in Lemma 1.2.

Prof. Ostrover and I, we want to go on to find an answer for this question.

1.5 Projected publications/articles resulting or to result from your grant

There is no publication planned so far.

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References

[1] Artstein-Avidan & Ostrover (2011), *Bounds for Minkowski Billiard Trajectories in Convex Bodies*. ArXiv e-prints, arXiv:1111.2353 [math.SG], November 2011, http://adsabs.harvard.edu/abs/2011arXiv1111.2353A.