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A scientific report concerning the short visit grant 5419 "Effective continuous infinitary logic" within the ESF activity entitled "New Frontiers of Infinity: Mathematical, Philosophical and Computational Prospects"

1. Purpose of the visit. The purpose of the 3 days visit (06 - 08.01.2013) hosted by Professor Sy-David Friedmann (Kurt Gödel Research Center, Wien, Austria) was investigations concerning logic actions of Polish groups on spaces of continuous structures, possible continuous versions of the Lopez-Escobar theorem on Borel invariant sets and effective aspects of continuous model theory.

2. Description of the work carried out during the visit.
(a) Talk at the KGRC seminar "**Polish G-spaces similar to logic G-spaces of continuous structures**" (a joint work together with Prof. Aleksander Ivanov, University of Wrocław), January 7, 2013;
(b) Discussions concerning nice topologies of the space of continuous structures and the role of possible extensions of the Lopez-Escobar theorem;
(c) Discussions concerning effectively presented continuous structures and effective continuous model theory of infinitary logic.

3. Description of the main results obtained. The visit and discussions with the members of the KGRC group became crucial for results obtained immediately after the visit. They were obtained together with Aleksander Ivanov. I briefly describe them below.

Let $(\mathbf{Y}, d)$ be a Polish space and $Iso(\mathbf{Y}, d)$ be the corresponding isometry group endowed with the pointwise convergence topology. Then $Iso(\mathbf{Y}, d)$ is a Polish group. It is worth noting that any Polish group G can be realised as a closed subgroup of the isometry group $Iso(\mathbf{Y}, d)$ of an appropriate Polish space $\mathbf{Y}$.

For any countable continuous signature L the set $\mathbf{Y}_L$ of all continuous metric L-structures on $(\mathbf{Y}, d)$ can be considered as a Polish $Iso(\mathbf{Y}, d)$-space. Let us call this action *logic*. The topology τ of this space is defined in a natural way (for example as in [U]). The paper [BYU] contains basic information concerning continuous model theory.

Note that for any tuple $\bar{s} \in \mathbf{Y}$ the map $g \to d(\bar{s}, g(\bar{s}))$ can be considered as a graded subgroup of $Iso(\mathbf{Y}, d)$. For any continuous sentence ϕ we have a graded subset of $\mathbf{Y}_L$ defined by $M \to \phi^M$. Some preliminaries concerning graded subsets and subgroups can be found in [BYM]

The main idea of our approach is as follows. Given Polish G-space $\mathbf{X}$ we prove that distinguishing an appropriate family of graded subgroups of G and some family $\mathcal{B}$ of graded subsets of $\mathbf{X}$ (called a graded nice basis) we arrive at the situation very similar to the logic space $\mathbb{U}_L$, where $\mathbb{U}$ is the bounded Urysohn space. Then treating elements of $\mathcal{B}$ as continuous formulas we together with Aleksander Ivanov obtain topological generalisations of several theorems from logic, for example Ryll-Nardzewski's theorem. The material below is a crucial element of this approach.

Logic actions over the Urysohn space

The construction of nice bases naturally arises when one considers the case of continuous logic actions over $\mathbb{U}$, the Urysohn space of diameter 1. This is the unique Polish metric space which is universal and ultrahomogeneous, i.e. every isometry between finite subsets of $\mathbb{U}$ extends to an isometry of $\mathbb{U}$. The space $\mathbb{U}$ is considered in the continuous signature $\langle d \rangle$.

The countable counterpart of $\mathbb{U}$ is the *rational Urysohn space of diameter 1* $\mathbb{QU}$, which is both ultrahomogeneous and universal for countable metric spaces with rational distances and diameter ≤ 1. It is shown in Section 5.2 of [BYBM] that there is an embedding of $\mathbb{QU}$ into $\mathbb{U}$ so that:

(i) $\mathbb{QU}$ is dense in $\mathbb{U}$;

(ii) any isometry of $\mathbb{QU}$ extends to an isometry of $\mathbb{U}$ and $Iso(\mathbb{QU})$ is dense in $Iso(\mathbb{U})$;

(iii) for any $\varepsilon > 0$, any partial isometry h of $\mathbb{QU}$ with domain $\{a_1, ..., a_n\}$ and any isometry g of $\mathbb{U}$ such that $d(g(a_i), h(a_i)) < \varepsilon$ for all i there is an isometry $\hat{h}$ of $\mathbb{QU}$ that extends h and is such that for all $x \in \mathbb{U}$, $d(\hat{h}(x), g(x)) < \varepsilon$.

We now define a family of clopen graded subgroups of $Iso(\mathbb{U})$ which satisfy the conditions of the theorem on existence of a nice topology which we proved together with Aleksander Ivanov. Let G_0 be a dense countable subgroup of $Iso(\mathbb{QU})$.

Family $\mathcal{R}$. Let $\mathcal{R}_0$ be the family of all clopen graded subgroups of $Iso(\mathbb{U})$ of the form

$$H_{q,\bar{s}} : g \to q \cdot d(g(\bar{s}), \bar{s}), \text{ where } \bar{s} \subset \mathbb{QU}, \text{ and } q \in \mathbb{Q}^+.$$

It is clear that $\mathcal{R}_0$ is closed under conjugacy by elements of G_0. Consider the closure of $\mathcal{R}_0$ under the operator *max* and define $\mathcal{R}$ to be the family of all

G_0-cosets of graded subgroups from $max(\mathcal{R}_0)$. Then $\mathcal{R}$ is countable and the family of all cones $(H_{q,\bar{s}})_{<l}$ where $H \in \mathcal{R}_0$ and $l \in \mathbb{Q}$, generates the topology of $Iso(\mathbb{U})$. Moreover the family satisfies the following property:

> for every graded subgroup $H \in \mathcal{R}$ and every $g \in G$ if $Hg \in \mathcal{R}$, then $H^g \in \mathcal{R}$.

We together with Aleksander Ivanov have proved that when a Polish group G has a family $\mathcal{R}$ as above, then the following statement holds.

Let $\langle \mathbf{X}, \tau \rangle$ be a G-space and $\mathcal{F}$ be a countable family of Borel graded subsets of $\mathbf{X}$ generating a topology finer than τ such that each $\phi \in \mathcal{F}$ is invariant with respect to some graded subgroup $H \in \mathcal{R}$.
*Then there is an $\mathcal{R}$-**nice topology** $\mathbf{t}$ for $(\langle \mathbf{X}, \tau \rangle, G)$ such that $\mathcal{F}$ consists of open graded subsets.*

The latter means that:
(a) The topology $\mathbf{t}$ is Polish, $\mathbf{t}$ is finer than τ and the G-action remains continuous with respect to $\mathbf{t}$;
*(b) There exists a graded basis $\mathcal{B}$ of $\mathbf{t}$ which is **nice with respect to** $\mathcal{R}$ in the following sense.*
(i) $\mathcal{B}$ is countable and rational cones of its elements generate $\mathbf{t}$;
(ii) for all $\phi_1, \phi_2 \in \mathcal{B}$, the functions $\neg\phi_1$, $min(\phi_1, \phi_2)$, $max(\phi_1, \phi_2)$, $|\phi_1 - \phi_2|$, $\phi_1 \dot{-} \phi_2$ $\phi_1 \dot{+} \phi_2$ belong to $\mathcal{B}$;
(iii) for all $\phi \in \mathcal{B}$ and $q \in \mathbb{Q}^+$ the doted multiplication $q \cdot \phi$ belongs to $\mathcal{B}$;
(iv) for all $\phi \in \mathcal{B}$ and open graded subsets $\rho \in \mathcal{R}$ we have $\phi^{*\rho}, \phi^{\Delta\rho} \in \mathcal{B}$ where

$$\phi^{\Delta\rho}(x) = inf\{r \dot{+} s : \{h : \phi(h(x)) < r\} \textit{ is not meagre in } \rho_{<s}\}.$$

$$\phi^{*\rho}(x) = sup\{r \dot{-} s : \{h : \phi(h(x)) \leq r\} \textit{ is not comeagre in } \rho_{<s}\};$$

(v) for any $\phi \in \mathcal{B}$ there exists an open graded subgroup $H \in \mathcal{R}$ such that ϕ is H-invariant.

Let L be a language of a continuous signature with inverse continuity moduli $\leq id$. Let $\mathbb{U}_L$ be the $Iso(\mathbb{U})$-space of all L-structures. The topology τ of this space is defined as above.

Let $\mathcal{B}_0$ be the family of all graded subsets defined by continuous L-sentences as follows

$$\phi(\bar{s}) : M \to \phi^M(\bar{s}), \text{ where } \bar{s} \in \mathbb{Q}\mathbb{U}.$$

It is easy to see that for any continuous sentence $\phi(\bar{s})$ there is a number $q \in \mathbb{Q}$ such that the graded subset as above is $H_{q,\bar{s}}$-invariant. The following theorem is the main result of our investigation.

Theorem 1. *The family $\mathcal{B}_0$ is an $\mathcal{R}$-nice basis.*

This can be considered as a version of Theorem 1.10 of [B] which states that in the discrete case of the S_∞-space of countable L-structures all formulas as above already form a nice basis. We prove that graded subsets associated with continuous formulas form a nice basis on $\mathbb{U}_L$. We call it 'logical' and consider it as the most natural example of nice bases.

The proof strongly depends on some preliminary work where we evaluate the distance between some types in $S_n(Th(\mathbb{U}))$. It can be also considered as a new amalgamation property of the Urysohn space.

Corollary 1.13 of [B] states that in the case of logic actions of S_∞ on the space of countable structures each nice topology is defined by model sets $Mod(\phi)$ of formulas of a countable fragment of $L_{\omega_1\omega}$. **Can this statement be extended to $\mathbb{U}_L$?**

We now give an example which shows that sraightforward generalisations do not hold. For this we introduce another family of graded subgroups.

Family $\mathcal{R}^{\surd}$. We now define an extension of $\mathcal{R}$. Let $\mathcal{R}_0^{\surd}$ be the extension of $\mathcal{R}_0$ by graded subsets of the form

$$H^{\surd}_{q,\bar{s}} : g \to q \cdot \sqrt{d(g(\bar{s}), \bar{s})}, \text{ where } \bar{s} \subset \mathbb{Q}\mathbb{U}, \text{ and } q \in \mathbb{Q}^+.$$

To see that graded sets of this form are graded subgroups take $g_1, g_2, g_3 \in Iso(\mathbb{U})$ with $g_1 \cdot g_2 = g_3$. Since all g_i are isometries, $d(g_1 g_2(\bar{s}), \bar{s}) \leq d(g_1(\bar{s}), \bar{s}) + d(g_2(\bar{s}), \bar{s})$. Thus $\sqrt{d(g_1 g_2(\bar{s}), \bar{s})} \leq \sqrt{d(g_1(\bar{s}), \bar{s})} + \sqrt{d(g_2(\bar{s}), \bar{s})}$ which implies the required inequality. When we apply max to a finite family of graded subgroups we obviously obtain a graded subgroup too. Let $\mathcal{R}^{\surd}$ be the family of all G_0-cosets of graded subgroups from $max(\mathcal{R}_0^{\surd})$. It is clear that it satisfies the corresponding assumptions of the existence theorem given above.

Theorem 2. *Let L be the language corresponding to the continuous signature $\langle d, c \rangle$, where c is a constant symbol. Then the family $\mathcal{B}_0$ of graded subsets of $\mathbb{U}_L$ defined above does not form an $\mathcal{R}^{\surd}$-nice basis.*

This shows that there are cases when 'logical bases' do not exist. Note that by the existence theorem the family $\mathcal{B}_0$ can be also extended to an $\mathcal{R}\surd$-nice basis of the G-space $\mathbb{U}_L$. Let $\mathcal{B}\surd$ be a basis of this form. We view it as the best example of a 'non-logical' basis.

In fact we show that $\mathcal{B}\surd$ cannot be even defined by continuous $L_{\omega_1\omega}$-formulas (defined in [BYIov]). Since this issue is strongly connected with the Lopez-Escobar theorem on invariant Borel subsets of logic spaces, we study possible versions of this theorem too.

The Lopez-Escobar theorem states that in the (discrete) S_∞-space of countable structures any invariant Borel subset is defined by a formula of $L_{\omega_1\omega}$. This theorem is crucial for Corolorary 1.13 of [B]. We show that the straightforward version of this theorem does not hold in the space of continuous structures.

Let $L = \langle d, R^1, c\rangle$ be a continuous signature where c is a constant symbol and R is a symbol of a predicate with continuity modulus id. Consider the logic space $\mathbb{U}_L$ of all continuous L-structures on the Urysohn space $\mathbb{U}$. To each continuous L-structure $M \in \mathbb{U}_L$ we associate a real number r_M defined as follows:

$$M \to r_M = sin(\frac{1}{R(c)^M}).$$

We assume that $r_M = 0$ for $R(c)^M = 0$. Since $R(c)$ is a formula and $sin(\frac{1}{x})$ is continuous in $(0,1]$, the function $M \to r_M$ is Borel. It is obviously $Iso(\mathbb{U})$-invariant.

Theorem 3. *There is no $L_{\omega_1\omega}$-sentence ψ so that for any $M \in \mathbb{U}_L$ the numbers r_M and ψ^M are the same.*

References

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4. Future collaborations with host institution. I expect to collaborate with Prof. Sy-David Friedman in investigations of effectively presented continuous structures by means of effective continuous model theory of infinitary logic. In particular I am planning to visit KGRC in October 2013.

5. Projected publications/articles resulting or to result from the grant. I am preparing the paper "Continuous Polish group actions arising in continuous logic" (together with Aleksander Ivanov).

6. Other comments. ———