Scientific report on ESF short visit grant 5472: Giant components in random hypergraphs

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Purpose of the visit

I visited Mihyun Kang at the Graz University of Technology from 10th till 23rd of March 2013 with the purpose of establishing collaboration on the project about emergence of giant components in random hypergraphs described below. During my stay I worked with Mihyun Kang and Oliver Cooley (also from TU Graz).

Description of the work carried out during the visit and the main results

The Erdős-Rényi random graph model [5] $G(n,p)$(resp. $G(n,M)$) is one of the most intensely studied in the theory of random graphs. It is well-known and has been studied in great detail (see e.g. [3, 6]) how the structure of the components changes as p (resp. M) grows. For example, there happens a sudden jump from when there are only components of size $O(\log n)$ to when a giant component of size $\Theta(n)$ in the random graph $G(n,p)$ emerges. This happens at the threshold $p = \frac{1}{n}$ (resp. $M = n/2$), and there are even more precise results known (the so-called sub- and supercritical regimes and the critical window). Also, similar phenomena were discovered in random hypergraphs. In particular, a straightforward generalization of the giant component was studied in [9, 7, 2, 1].

While in the graph case two vertices are connected if there is a path (or walk) between them, in the hypergraphs the notion of a path (or walk) is ambiguous and in fact there are several possible definitions. In the cases mentioned above the following notion was studied: two vertices u and v are connected in a k-uniform hypergraph if there is a sequence of hyperedges $h_0,\dots,\ h_n$ such that $u \in h_0$ and $v \in h_n$ and $h_i \cap h_{i+1} \neq \emptyset$. The threshold for the random k-uniform hypergraph was first determined in [9] to be $p = \frac{(k-2)!}{n^{k-1}}$. Furthermore, at this density the edges h_i typically intersect in at most one vertex.

We proposed to study the following notion of j-tuple connectivity in k-uniform hypergraphs. We say that two j-sets J_0 and J_n ($j \in \{1,\dots,k-1\}$) are connected in the k-uniform hypergraph H if there is an alternating sequence of j and k-element subsets of $V(H)$: $J_0, h_0, J_1, h_1, \dots, J_n$ such that $J_i \cup J_{i+1}$ is contained in h_i, where h_is are the hyperedges of H. The components then consist of j-element subsets of the vertex set of H. Again, one might wonder when a j-tuple connected giant component of size $\Theta(n^j)$ emerges in the random k-uniform hypergraph. The intuition from the theory of branching processes suggests this to happen above

$$p_{k,j} = p_{k,j}(n) := \frac{(k-j)!j!}{(k)_j - j!} n^{j-k}. \tag{1}$$

Observe that $p_{k,1}$ gives precisely the threshold for vertex connectivity in random (hyper-)graphs.

Our approach builds on the recent proof strategy of Krivelevich and Sudakov [8] who used the depth-first search algorithm in graphs to give a simple and short proof of the phase transition in random graphs. More precisely, we adapted their proof strategy and looked first at the phase transition in random hypergraphs for $j = 1$. This yields alternative proofs for the emergence of giant component studied before giving the right order of magnitude of the giant. Moreover, the approach via depth-first search allows to study the giant component in the supercritical phase, i.e. when $p = (1+\varepsilon)\frac{(k-2)!}{n^k}$ with $\varepsilon = \varepsilon(n) \gg \sqrt[3]{\log n/n}$, which gives the right lower bound on the size of the giant component in random hypergraphs giving an alternative proof of one of the results obtained first by Karoński and Łuczak in [7].

Then we looked at the first unknown case $j = 2$ and $k = 3$, where two 2-element sets J and J' of the vertex set $[n]$ are connected in the 3-uniform hypergraph $H = ([n], E)$ if there is a sequence $J = J_0, h_0, J_1, h_1, \ldots, J_n = J'$ such that $J_i \cup J_{i+1} = h_i$, where h_is are the hyperedges of H.

Our work in progress gives the following theorem thus confirming the threshold for $p_{3,2}$ mentioned above.

Theorem 1. *Let $\varepsilon > 0$ be given. Then a.a.s. the size of the 2-tuple connected largest component in the random 3-uniform hypergraph $H^3(n,p)$ is $\Omega(\varepsilon n^2)$ if $p = \frac{1+\varepsilon}{2n}$ and $O(\frac{\log n}{\varepsilon^2})$ if $p = \frac{1-\varepsilon}{2n}$. Moreover, for $p = \frac{1+\varepsilon}{2n}$, the giant component contains all vertices a.a.s.*

While the case of $O(\log n)$ large components follows more or less directly from the depth-first search approach, the case of the giant component requires some more elaborate analysis involving concentration inequalities for martingales and binomial branching processes. It seems possible to extend the theorem above for k-uniform hypergraphs. Furthermore, one should be able to study the supercritical phase as well.

Future collaboration and projected publications

The results described above are currently being written up [4]. Future exchange visits are intended to further study the above problem and work on the completion of the paper. The European Science Foundation and the program "Random Geometry of Large Interacting Systems and Statistical Physics" (RGLIS) will be gratefully acknowledged in any published and/or presented work resulting from this collaboration.

I greatly enjoyed the scientific environment provided by my host Prof. Mihyun Kang at the Technische Universität Graz.

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