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Scientific Report on Exchange Visit No. 5428

Host Academic: Prof. Dr. Andreas Weiermann
Host Institution: University of Ghent, Belgium
Visitor: Dr. Peter Schuster, University of Leeds, UK
Project Title: *Finite from Transfinite Proofs*

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Project Work During this visit, the project theme *Finite from Transfinite Proofs* was studied along the following lines, which are part of joint work with Ulrich Berger (Swansea University, UK). To explain the progress that has been made during the visit, we now look at the following statement from abstract linear algebra, which indeed is a paradigmatic case:

> *Let V be a vector space and $v \in V$. If $\phi(v) = 0$ for all linear forms on V, then $v = 0$.*

The typical proof of this assertion is indirect and makes use of Zorn's Lemma (ZL), as follows. Suppose that $\phi(v) = 0$ for all linear forms on V. Assume toward a contradiction that $v \neq 0$. Invoking ZL, extend v to a basis of V. Define a linear form ϕ on V by setting $\phi(v) = 1$ and $\phi(w) = 0$ for every basis vector $w \neq v$. Contradiction!

This proof is anything but constructive, but the statement itself has very concrete applications: its generalisation to continuous linear functionals on a locally convex topological vector space "...is the basis of a standard method of treating certain approximation problems" [5, p. 59, Remark]. It therefore makes sense to attempt an "unwinding" of the proof.

To this end, we move to an even slightly more abstract level. In fact the statement under consideration is one of the many that follow the very same pattern, as follows:

> *Let X be the directed-complete partial order that is formed by certain substructures U of a given algebraic structure V. Let $v \in V$. If $v \in U$ for all maximal $U \in X$, then $v \in U$ for all $U \in X$.*

(To get back the statement from linear algebra we have started from, let X consist of the proper subspaces of the vector space V, and notice that the maximal among those are just the hyperplanes, i.e. the kernels of non-zero linear forms.) As is known from previous work also by the visitor [3, 6], the key to the computational content of such proof is to make it direct, replacing ZL in the form

> *Let X be a directed-complete partial order, and let $C \subseteq X$ be closed under directed joins. If $C \neq \emptyset$, then C has a maximal element.*

by Raoult's Open Induction (OI) [1, 2, 4]: that is, *Let X be a directed-complete partial order, and let $O \subseteq X$ split directed joins. If O is downwards progressive, then $O = X$.*

While in [3, 6] this transformation was done in a fairly ad hoc way, it has turned out during the visit that a more systematic approach is possible—and that there is only one crucial step, which can be distinguished clearly. We now briefly sketch this development.

In a typical indirect use of ZL as the one above, and with the terminology from before, C is the complement of O. (In our example O consists of the proper subspaces U with $v \in U$, and O of the proper subspaces U with $v \notin U$.) Hence the natural way to prove that C is closed under directed joins is to prove that O splits directed joins. This is easy to see.

The second main ingredient, however, is the tricky one: the argument that eventually leads to the desired contradiction. Typically this reads as follows:

(†) *If x is a maximal element of C, then x belongs to O.*

(The contradiction now arises because C is the complement of O.)

The major step forward we have made is the insight that to prove (†) amounts to prove that $\neg\neg O$, the double complement of O, is downwards progressive. As it is unfeasible to assume that O is stable (normally O is an existential statement), the next item on our agenda is to understand under which circumstances, and how, a proof of the downwards progressivity of $\neg\neg O$ can be reduced to a proof of the downwards progressivity of O itself.

Dissemination On the afternoon of Thursday 4th February 2014, the visitor has given a one-hour talk at the host institution, on *Folding Up (Disjunctions)*. Here is the abstract:

> Part of the understanding of disjunction in a model of classical logic is the implication (*) *if $A \vee B$ holds, then A holds or B holds.* Using (*) in a mathematical proof means to open up two branches in the proof tree in which one knows more: A holds in one and B in the other. The sharper information normally makes it easier to proceed with the proof. The characteristic axiom of many an ideal object in algebra and logic (prime ideal or filter, dichotomy of order, complete theory etc.) can be put as a variant of (*), with an appropriate operation in place of disjunction. Apart from individual instances this is little known, albeit little surprising from the perspective of formal topology: once completed to an equivalence, (*) says that the operation witnesses the intersection of basic opens or, equivalently, the filtering property of formal points. Typically the characteristic axiom can be eliminated, or at least reduced to the special case that corresponds to (*) with $A = B$. The special case is not always trivial for operations other than disjunction. Anyway, the branchings created by uses of (*) disappear from the proof trees. The reduction is usually proved by invoking an appropriate form of the Axiom of Choice (Krull's Lemma, the Artin-Schreier Theorem, Gödel's Completeness Theorem etc.). As a consequence, the proof of the reduction falls short of telling us why it works, and why $A = B$ matters. In spite of the transfinite odour around the reduction, we aim at equally getting by with finite methods only, as it already works in some instantiations. We expect to thus get reasons and routines on top of the facts.

Further Interaction Apart from his daily interaction with the host academic, the visitor has used the opportunity to exchange ideas with Jeroen Van der Meeren (doctoral student), Paul Shafer (postdoctoral fellow), Lev Gordeev (research fellow) and Alberto Marcone (visiting Ghent from the Università degli Studi di Udine, Italy). In particular, it has turned out that Gordeev's new technique of compressing tree representations into representations by directed acyclic graphs, can be instantiated in the visitor's own recent work [6].

Side Effects For some hours during the visit the visitor joined his coauthor Riccardo Bruni (Università degli Studi di Firenze, Italy) in finalising the revised version of the paper "Approximating Beppo Levi's *principio d'approssimazione*", which is under consideration for publication in the *Bulletin of Symbolic Logic*.

References

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