REPORT ON SHORT VISIT 5427 OF ESF INFTY

SAM SANDERS

1. Purpose of the visit

The purpose of my visit to the *Department of Mathematics* of *Stockholm University* was to undertake joint research with Erik Palmgren and his group on the connection between computability, constructive mathematics and Nonstandard Analysis. Erik Palmgren and his group are well-known for their research in both these areas, while I have been working on the connection between these areas for some years now.

2. Description of the work carried out during the visit

The mathematical practice of Nonstandard Analysis takes the following *simplified* form. The mathematics in the nonstandard world is usually very 'constructive', in the sense that it is just explicit algebraic calculations without any use of the (nonconstructive) law of excluded middle ([11]).

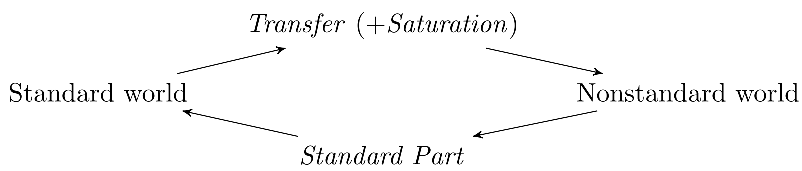

Figure 1. The practice of NSA

The principles *Transfer* and *Standard Part* of Nonstandard Analysis ([6]) are however well-known to give rise to non-computable objects ([2, 4]). During my stay, Erik Palmgren and me obtained optimal 'computable' versions of both Transfer and Standard Part. We investigated if these principles give rise to natural results in Reverse Mathematics ([10]), especially higher-order Reverse Mathematics à la Ulrich Kohlenbach ([3]).

3. Description of the main results obtained

In [5], the authors introduce Ω-invariance, an equivalent definition of Turing computability based on infinitesimals from Nonstandard Analysis. Intuitively, an object is Ω-invariant if it is *independent of the choice of infinitesimal.* More formally, we have the following definition, where Ω is the set of infinite integers.

1. **Definition** (Ω-invariance). *Let $f^{(0\times 0)\to 0}$ be standard and M^0 be infinite. The function $f(n^0, M)$ is Ω-invariant if*

$$(\forall^{st} n)(\forall N^0 \in \Omega)[f(n, M) =_0 f(n, N)]. \tag{1}$$

The goal of my visit to Palmgren was to extend the notion of Ω-invariance to higher types, as in its above form it is limited to low types. The ideal setting for this is the higher-type systems HA^ω, iHA^ω, and RCA_0^ω ([1, 3, 4]). We were successful in that we obtained a uniform definition for Ω-invariance in higher types as can be found below in Definition 3. We also obtained ($\bigstar$), a version of the Transfer principle as in Principle 4 which results in RCA_0^Ω, a conservative extension of RCA_0^ω. In general, we have shown/obtained that

(1) The general definition from Definition 3 reduces to the particular one from Definition 1.
(2) The computable Standard Part principle (provable in RCA_0^Ω): For every Ω-invariant hyperrational q_ω, there is $x \in \mathbb{R}$ such that $x \approx q$.
(3) The Reverse Mathematics of $(\exists^2)$ can be obtained easily using Ω-invariance and ($\bigstar$) in the base theory RCA_0^Ω.
(4) For every uniform version of a principle equivalent to $(\exists^2)$, there is a non-uniform nonstandard version obtained by applying Transfer to the inner-most universal formula.
(5) The non-uniform nonstandard versions form the previous item, all come from ERNA's Reverse Mathematics ([7]).

By the final item, we have shown that Higher-order Reverse Mathematics unifies Friedman-Simpson style Reverse Mathematics ([10]) and ERNA's Reverse Mathematics ([7–9]).

We now list some of the technical definitions, for completeness. Here, $\text{St}(x)$ is the usual standardness predicate from Nonstandard Analysis which is defined by the usual axioms in RCA_0^Ω.

First of all, in RCA_0^ω, for every finite type ρ, the associated equality $=_\rho$ is defined as follows (See [3, §2]): For x^ρ, y^ρ, the formula $x =_\rho y$ is short for

$$(\forall z_1^{\rho_1} \dots z_k^{\rho_k})\big[x z_1 \dots z_k =_0 y z_1 \dots z_k\big], \tag{2}$$

for $\rho = [\rho_1 \to \dots \to \rho_k \to 0]$. In other words, two objects of the same type are *equal* if they produce equal outputs on equal inputs of lower type, down to type 0.

2. **Definition.** For $\rho = [\rho_1 \to \dots \to \rho_k \to 0]$, we define *approximate equality* $x^\rho \approx_\rho y^\rho$ as $(x =_\rho y)^{st}$, i.e. (2) with 'st' affixed to every quantifier.

With this definition, two objects of the same type are approximately equal if they produce equal outputs on equal *standard* inputs of lower type, all the way down to type 0. The definition of Ω-invariance is then as follows.

3. **Definition.** [Ω-invariance] Let $f^{(\tau\times 0)\to\rho}$ be standard and fix $m^0 \in \Omega$. The function $f(x^\tau, m^0)$ is *Ω-invariant* if $(\forall^{st} x^\tau)(\forall m^0, n^0 \in \Omega)(f(x^\tau, m^0) \approx_\rho f(x^\tau, n^0))$.

Secondly, in the Robinsonian approach to Nonstandard Analysis, a set $A = \{n \in \mathbb{N} : \varphi(n)\}$ from the standard universe is extended to ${}^*A = {}^*\{n \in \mathbb{N} : \varphi(n)\} = \{n \in {}^*\mathbb{N} : {}^*\varphi(n)\}$ via the star morphism. In other words, if A is characterized by φ, then ${}^*A$ is characterized by ${}^*\varphi$. The following principle ($\bigstar$) captures this characterization for functions in the context of RCA_0^Ω: If a standard function F is characterized on the standard objets via a formula A^{st} (A internal), then A should characterize F on all (standard and nonstandard alike) objects. Alternatively, one can see ($\bigstar$) as a special case of the transfer principle of Nonstandard Analysis where one has to 'justify' the instance of transfer $A^{st} \to A$ by providing a standard function deciding A^{st} first.

4. **Principle ($\bigstar$).** *For standard $F^{\tau\to 0}$ and internal $A(x^\tau)$, we have*

$$(\forall^{st} x^\tau)[F(x) = 1 \leftrightarrow A^{st}(x)] \to (\forall x^\tau)[F(x) = 1 \leftrightarrow A(x)]. \tag{3}$$

4. Future collaboration with host institution

Currently, Erik Palmgren and I are writing up the aforementioned results in a joint paper. In the future, I will visit Stockholm again to work on follow-up papers regarding e.g. the Suslin operator and Horst Osswald's notion of *local constructivity*.

In February, the group of Prof. Kazuyuki Tanaka (Tohoku University) organizes an international annual workshop on Computability Theory and the Foundations of Mathematics. For the 2014 iteration, Erik Palmgren has agreed to be an invited speaker. Finally, for the honorary PhD which Harvey Friedman will obtain from Ghent University in September 2013, Erik Palmgren and Per Martin-Löf have tentatively agreed to visit Ghent during that period.

5. Projected publications

As mentioned before, Erik Palmgren and I are working on a joint paper on the connection between Nonstandard analysis and Higher-order Reverse Mathematics, specifically documenting the results regarding nonstandard RCA_0^ω and $(\exists^2)$.

We believe at least a couple of additional papers on this topic will follow from this visit.

6. Other comments

Stockholm is a rather expensive town, say compared to Munich or Belgium in general. Perhaps it is a good idea to make the per diem dependent on the place one is visiting, if this does not cause too much overhead or international strife due to chauvinism.

References

[1] Errett Bishop, *Foundations of constructive analysis*, McGraw-Hill Book Co., New York, 1967.
[2] H. Jerome Keisler, *Nonstandard arithmetic and reverse mathematics*, Bull. Symbolic Logic **12** (2006), no. 1, 100–125.
[3] Ulrich Kohlenbach, *Higher order reverse mathematics*, Reverse mathematics 2001, Lect. Notes Log., vol. 21, Assoc. Symbol. Logic, La Jolla, CA, 2005, pp. 281–295.
[4] Ieke Moerdijk and Erik Palmgren, *Minimal models of Heyting arithmetic*, J. Symbolic Logic **62** (1997), no. 4, 1448–1460.
[5] Antonio Montalbán and Sam Sanders, *Nonstandard Analysis and Turing Computability*, In preparation (2012).
[6] Abraham Robinson, *Non-standard analysis*, North-Holland, Amsterdam, 1966.
[7] Sam Sanders, ERNA *and Friedman's Reverse Mathematics*, J. of Symb. Logic **76** (2011), 637-664.
[8] Sam Sanders and Keita Yokoyama, *The Dirac delta function in two settings of Reverse Mathematics*, Archive for Mathematical Logic **51** (2012), no. 1, 99-121.
[9] Sam Sanders, *Reverse-engineering Reverse Mathematics*, To appear in Annals of Pure and Applied Logic, CiE2011 Special Issue (2012).
[10] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, Cambridge, 2009.
[11] F. Wattenberg, *Nonstandard analysis and constructivism?*, Studia Logica **47** (1988), 303–309.