SCIENTIFIC REPORT

for ESF Short Visit Grant Application 5913 (INFTY)

The aim of the visit was to extend the collaboration between Sam Sanders and myself on a project on connections between non-standard mathematics and constructive analysis, viewed through the eyes of Ω-invariance.

In forthcoming work, Montalbán and Sanders (2013) have shown that Ω−CA (a comprehension principle stating that Ω-invariant constructions yield "real" objects) can be added to weak systems of classical non-standard analysis to yield conservative extensions. Here, an Ω-invariant construction is a standard function $F\colon {}^*A \times {}^*\mathbb{N} \to {}^*B$ such that for any two infinite $\omega, \omega'\colon {}^*\mathbb{N}$, $F(u,\omega) = F(u,\omega')$. Then by Ω−CA, there is a standard function $G\colon {}^*A \to {}^*B$ such that $F(u,\omega) = G(u)$ for any (and all) infinite $\omega\colon {}^*\mathbb{N}$.

Prior to my visit, Erik Palmgren and Sam Sanders had established that Ω−CA is true in the filter model of constructive nonstandard analysis (Palmgren 1998). During my visit, Sam and I investigated Ω−CA and variants for other constructive systems of non-standard analysis, such as those for Martin-Löf (1990), Moerdijk (1995), and Palmgren (1995). We didn't reach a definite understanding of this issue yet, but it seems that Ω−CA is conservative over all of the base systems.

A central remaining question is whether the principle ($\mathbb{M}$) is conservative over RCA_0^Ω. This principle states, for $\Phi(x) \in \Pi^0_1$ and standard x^0, y^0, if

$$\big(\Phi(x) \wedge (\forall z < x.\, \neg\Phi(z))\big) \wedge \big(\Phi^{\mathrm{st}}(y) \wedge (\forall z < y.\, \neg\Phi^{\mathrm{st}}(z))\big),$$

then $x = y$. This expresses that (Π^0_1)- and $(\Pi^0_1)^{\mathrm{st}}$-minimizations agree. This seems very plausible but awaits confirmation.

During my visit, I updated Sam on the technical aspects of Feferman's systems of explicit mathematics (Feferman 1975). There's a reasonable chance that the might provide a good setting for studying non-standard analysis in a type-free setting, in contrast to the usual approaches.

We still hope that this work will yield dividends for the constructive understanding of the infinite and infinitesimal; provide new results in the reverse mathematics of physically applicable mathematics; and extend the nascent interpretation of the constructive world in terms of the infinite.

We did not obtain any definite results during the visit, but we will be continuing our collaboration on this project in the future. It's likely that our collaboration will result in one or more publications, but so far we do not have definite plans. We will be sure, however, to acknowledge the ESF in all publications arising from the work.

REFERENCES

Feferman, Solomon (1975). "A language and axioms for explicit mathematics". In: *Algebra and logic*. Vol. 450. Lecture Notes in Mathematics. Fourteenth Summer Res. Inst., Austral. Math. Soc., Monash Univ., Clayton, 1974: Springer, Berlin, pp. 87–139. DOI: 10.1007/BFb0062852.

Martin-Löf, Per (1990). "Mathematics of infinity". In: *COLOG-88 (Tallinn, 1988)*. Vol. 417. Lecture Notes in Comput. Sci. Berlin: Springer, pp. 146–197. DOI: 10.1007/3-_540-_52335-_9_54.

Moerdijk, Ieke (1995). "A model for intuitionistic non-standard arithmetic". In: *Ann. Pure Appl. Logic* 73.1. A tribute to Dirk van Dalen, pp. 37–51. ISSN: 0168-0072. DOI: 10.1016/0168-_0072(93)E0071-_U.

Montalbán, Antonio and Sam Sanders (2013). "Nonstandard Analysis and Turing Computability". In preparation.

Palmgren, Erik (1995). "A constructive approach to nonstandard analysis". In: *Ann. Pure Appl. Logic* 73.3, pp. 297–325. ISSN: 0168-0072. DOI: 10.1016/0168-_0072(94)00030-_7.

— (1998). "Developments in constructive nonstandard analysis". In: *Bull. Symbolic Logic* 4.3, pp. 233–272. ISSN: 1079-8986. DOI: 10.2307/421031.