SCIENTIFIC REPORT

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I was invited to Vienna by Dr. Jakob Kellner of the Kurt Gödel Research Centre, to resume my work with Dr. Misha Gavrilovich from last year. Due to scheduling conflicts I have missed the opportunity to work with Dr. Liuzhen Wu as I was hoping to.

In that week of work we focused on the continuation of our research from last year, which we have been discussing over emails as well. We came to realize that some of the proposed definitions and notions that we defined were a bit problematic, and we tried to remedy the problems that we faced. The main topic was to try and connect PCF theory with the language of homotopy and model categories.

1. Constructing Categories From Pre-Orders

Recall that a pre-ordered set is a pair $(P, \leq)$ such that $\leq$ is a relation on P which is reflexive and transitive.

Given a pre-ordered set $(P, \leq)$ we can construct the category $\mathrm{St}_{\leq}(P)$ whose objects are non-empty subsets of P, and given such F, G we define an arrow $F \to G$ if and only if for every $f \in F$ there exists $g \in G$ such that $f \leq g$.

We observe that if there is an arrow from F to G, two objects, then that arrow is unique. In some cases we are lucky enough that we can endow the arrows with an additional structure to make $\mathrm{St}_{\leq}(P)$ a model category (or rather, almost a model category, as we may need to restrict to a particular subclass of objects, or add more arrows).

If $(P, \leq)$ and $(Q, \preceq)$ are two pre-ordered sets, we say that a function $\pi\colon P \to Q$ is continuous if $f \leq g \implies \pi(f) \preceq \pi(g)$; and whenever $f \wedge g$ $(f \vee g)$ exists in P, we have that

$$\pi(f \wedge g) = \pi(f) \wedge \pi(g)$$

(and similarly for $\vee$). We can define a functor $\hat{\pi}\colon \mathrm{St}_{\leq}(P) \to \mathrm{St}_{\preceq}(Q)$ by $\hat{\pi}(F) = \pi''F$, and from now on we shall abuse the notation and write them both as π when the context allows it. Moreover if π was continuous to begin with then we shall say that $\hat{\pi}$ is continuous as well, so there is no confusion when we say that π is continuous in either case.

2. PCF Theory

In PCF theory one begins with a set A whose members are regular cardinals, and $|A| < \min A$. We ask, given an ideal I on A, is there a cofinality to the ordering $(\prod A, \leq_I)$ (dominance modulo I)? What are the possible cofinalities when we consider all the possible ideals over A?

2.1. **Exact upper bounds.** Given a family of functions $F = \{f_\alpha \mid \alpha < \lambda\} \subseteq \prod A$, we say that F has an exact upper bound f in $\leq_I$ if for every α, $f_\alpha \leq_I f$, and whenever g is such that $f_\alpha \leq_I g$ for all α, then $f \leq_I g$; and if whenever $g <_I f$ then for some α, $g <_I f_\alpha$.

Exact upper bounds are important in PCF theory, and the last property of exact upper bound, seems like a very good candidate to be a homotopy invariant. Which is the goal of our project.

2.2. **Bounding numbers of ideals.** Let I be an ideal on A, we define the following cardinal:

$$\mathfrak{b}(I) = \min\{\kappa \mid \exists\{f_\alpha \in \prod A \mid \alpha < \kappa\}\forall g \exists \alpha : f_\alpha \not\leq_I g\}.$$

We note that if I is a maximal ideal then $(\prod A, \leq_I)$ is a linear order and $\mathfrak{b}(I)$ is the cofinality of I.

One of the celebrated theorems of PCF tells us that under certain conditions there exists I_λ such that $\mathfrak{b}(I_\lambda) = \lambda$, and whenever $\mathfrak{b}(J) = \lambda$ we have that $I_\lambda \subseteq J$. This, similarly to the exact upper bound properties, is a good candidate for homotopy invariant.

3. Previous Work

In our previous work we attempted the following definition. We work in $\mathrm{St}(\mathrm{Ord}^A)$, where A is a set of ordinals, and we define $X \xrightarrow{(c)} Y$ if for every $y \in Y$ there exists $x \in X$ such that $y \leq_I x \wedge y$; and $X \xrightarrow{(wc)} Y$ if for every $y \in Y$ there is some $x \in X$ such that $y <_I x$.

The motivation for this definition is that if F, G are two sets in Ord^A which have the same exact upper bound then there is a (wc) arrow between them. The idea is that cofibrant objects are those equivalent to zero modulo I, and (wf) arrows are somehow model some sort of eventual dominance over every other object.

However this definition is problematic for two reasons. The foremost problem is that fibrant objects end up as proper classes which are somewhat useless in the common context of PCF theory, as we are interested in the product of cardinals and the possible cofinalities of these partially ordered sets.

The second reason is in some sense dual to the first. Cofibrant objects are sets of functions which are equivalent to the zero function modulo I. While this is a possibly useful notion, we are more interested in bounded subset than in zero subsets. The result is that we end up having too many cofibrant objects, and when we come to decompose an arrow to a (c) arrow followed by a (wf) arrow we run into the following problem. Given any object F, we have $\{0\} \to F$. This can be decomposed to the following diagram:

$$\begin{tikzcd}
\{0\} \arrow[rr] \arrow[dr, "(c)"'] & & F \\
& Y \arrow[ur, "(wf)"'] &
\end{tikzcd}$$

But this means that Y is equivalent to $\{0\}$ modulo I, and if (wf) arrows should somehow model dominance then F is dominated by Y which means it is also equivalent to 0 modulo I.

4. The New Definitions

After spending the first two days realizing our mistakes from a year ago, we decided to try a different angle, and instead tried to focus on bounded sets rather than zero objects.

In this report we relativize the definitions and ideas to the most striking use of basic PCF theory, but the definitions easily generalize.Our example is $A = \{\aleph_{n+1} \mid n \in \omega\}$, and the first and amazing use of PCF theory is to show that in A, when taking dominance modulo finite coordinates, there is a sequence of size $\aleph_{\omega+1}$.

In addition to that, we are interested in function which reach the diagonal of the product $\prod A$, and so we will consider $P = \prod_{n\in\omega}(\omega_{n+1}+1)$ with $\leq_I$, for an ideal I on ω. We shall denote by $\mathrm{St}_I(A)$ the category $\mathrm{St}_{\leq_I}(P)$, for the trivial ideal we simply write $\mathrm{St}(A)$.

Seeking out a model category structure over $\mathrm{St}_I(A)$ we need to find a suitable definition for labels, and we are looking for one which will behave nicely with both $\mathfrak{b}(I)$ and the notion of exact upper bounds (that is, we want that two families which have the same exact upper bound to be homotopically equivalent).

Definition 1. Suppose that F and G are objects in $\mathrm{St}(A)$ such that $F \to G$. We label the arrow as $(wc)_\lambda$ if the following holds:

For every continuous $\pi\colon (P, \leq) \to \lambda$, if $\pi(F)$ is bounded then $\pi(G)$ is bounded.

This brings up to the following conjecture about the characterization of $(wc)_\lambda$ arrows:

Conjecture 2. $F \to G$ is $(wc)_\lambda$ if and only if $F \to G$ in St_{I_λ}.

Unfortunately, crystalizing these definitions has left us with very little time to solve this conjecture, and advance further in our development of the dictionary translating PCF theory to a model categorical, and homotopy based languages. Much work is left to do, and we hope to continue it over email and on future occasions.

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