# Scientific Report: Giant Components in Random Hypergraphs Supported by ESF Short Visit Grant 5639

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## Purpose of the Visit

From 29th May to 4th June 2013 I visited the Free University of Berlin to continue a collaboration begun with Yury Person when he visited TU Graz on a previous ESF Short Visit Grant (5742) on the project described below.

## Description of the work carried out during the visit and the main results

The Erdős-Rényi model [5] $G(n,p)$ is perhaps the most well-known and well-studied random graph model, and many of its features and behaviours are by now very well understood. For example, the existence of a so-called **giant component** of order $\Theta(n)$ undergoes a **phase transition** at $p = 1/n$. More precisely, for any $\varepsilon > 0$, $G(n, \frac{1-\varepsilon}{n})$ with high probability does not contain a giant component, while $G(n, \frac{1+\varepsilon}{n})$ with high probability contains a unique giant component (see e.g. [1]). Phase transition behaviour is also observed in other fields, such as percolation theory and statistical physics.

While Yury Person was visiting Graz University of Technology during March 2013 on a previous ESF grant, together with Mihyun Kang we considered the generalisation of this problem to 3-uniform hypergraphs, in which there are two different possible generalisations of connectedness. Two pairs $f_1, f_2 \in \binom{V}{2}$ are *tightly connected* if there is a sequence of edges $e_1, \dots, e_\ell$ such that $|e_i \cap e_{i+1}| = 2 \ \forall 1 \le i \le \ell - 1$ and $f_1 = e_1$, $f_2 = e_\ell$. A giant component is a set of $\Theta(n^2)$ tightly connected pairs.

Inspired by the recent, elegant new proof of Krivelevich and Sudakov [6] of the phase transition in the graph case, we were able to prove [4] that the random 3-uniform hypergraph model $H^3(n,p)$ exhibits a phase transition for the emergence of a giant tight component at $p = 1/2n$. Furthermore, we were able to use similar methods to provide a new and simple proof of the analogous phase transition result for vertex connectivity, which had already been well studied, e.g. [7, 3, 2].

While visiting Yury Person at the Free University of Berlin, we first re-visited the proof of the phase transition for a giant tight component in the 3-uniform case, filling in many previously missing details. As a result we have now largely completed the writing up of the proof of the following result.

**Theorem 1.** *Given $\varepsilon > 0$, the size of the largest tightly connected component in the random 3-uniform hypergraph $H^3(n,p)$ is $\Theta(n^2)$ if $p = \frac{1+\varepsilon}{2n}$ and $O(\log n)$ if $p = \frac{1-\varepsilon}{2n}$.*

The main difficulty involved in generalising the techniques for graphs is that at various stages in the argument we require that every vertex is in some sense typical. More precisely we require that not too many pairs involving a vertex have already been visited by the depth first search. This corresponds to bounding the maximum degree of the graph of pairs visited.

We then moved on to consider how to generalise our proof strategy to $j$-connected components in $k$-uniform hypergraphs. For $1 \leq j \leq k-1$, two $j$-sets $f_1, f_2 \in \binom{V}{j}$ are *$j$-connected* if there is a sequence of edges $e_1, \ldots, e_\ell$ such that $|e_i \cap e_{i+1}| = j \;\; \forall 1 \leq i \leq \ell - 1$ and $f_1 = e_1,\ f_2 = e_\ell$. A $j$-connected component is a maximal set of $j$-connected $j$-sets. A giant component is component of size $\Theta(n^j)$. (Note that for $k=3$, the case $j=2$ corresponds to tight connectedness in 3-uniform hypergraphs as described above, while for $j=1$ this corresponds to vertex-connectedness.) For this general case we now also have an outline of a proof of the following theorem, which we are currently preparing for publication.

**Theorem 2.** *Given integers $j \leq k$, let $p_0 = p_0(n) := \frac{(k-j)!}{(\binom{k}{j}-1)n^{k-j}}$. Given $\varepsilon > 0$, the size of the largest $j$-connected component in the random $k$-uniform hypergraph $H^k(n,p)$ is $\Theta(n^j)$ if $p = (1+\varepsilon)p_0$ and $O(\log n)$ if $p = (1-\varepsilon)p_0$.*

The main difficulty in moving from the 3-uniform case to the general $k$-uniform case is that now rather than only needing to bound the degrees in a graph of visited pairs, we in fact need to bound the $i$-degree in the $j$-uniform hypergraph of visited $j$-sets for each $1 \leq i < j$. (In the case $k=3$ we had $j=2$, and therefore $i=1$ was the only case to consider.)

In both the 3-uniform and the general $k$-uniform case, the arguments involve a number of different techniques involving probabilistic methods, including **branching processes**, coupling of random variables, **martingales**, **dynamic processes** and depth first search trees. In both cases the giant component is unique by a "sprinkling" argument.

## Future collaboration and projected publications

The funding supplied by the ESF short visit grants 5742 and 5639 have allowed us to develop a collaborative project between Yury Person at the Free University of Berlin and Mihyun Kang and myself at Graz University of Technology. I have greatly enjoyed this joint work and we intend to continue working together, on this project and others, in the future. Several further questions on this project immediately suggest themselves, including what happens in the **critical window**, i.e. we consider the behaviour when $\varepsilon$ is a $o(1)$ function rather than a constant.

Our first aim is to finish writing up the results that we have already obtained, outlined above. As previously mentioned, the proof of the 3-uniform case is at an advanced stage. The general $k$-uniform, $j$-connected case is still in the early stages. However, it is expected that this proof will ultimately supercede the 3-uniform paper.

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# References

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