FInal report on the ESF-EMS-ERCOM conference Teichmüller Theory and its Interactions in Mathematics and Physics 28 June — 3 July 2010 Centre de Recerca Matemàtica, Bellaterra, Spain

Robert C. Penner & Athanase Papadopoulos

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1 Conference highlights

Teichmüller theory in a broad sense is the study of moduli spaces for geometric structures on surfaces, and it can be considreed from different points of view, making connections between several fields of mathematics, inluding low-dimensional topology, algebraic topology, hyperbolic geometry, representations of discrete groups in Lie groups, symplectic geometry, topological quantum field theory, string theory, and other fields of mathematical physics. This subject is growing at an exceptional rate and the conference was a forum for an ewposition and the discussion of the recent and new ideas in the theory.

We can honestly say that this was a spectacularly successful event with more than 85 participants from all over the world, many of them saying that it was among the best conferences of their lives in terms of scientific level, novelty, and inspiration. As opposed to expressing any immodesty on the part of the organizers, this is rather a statement about the caliber of the speakers and the timeliness of the event itself. A number of the presentations were on cutting-edge results which have only very recently been written up, so the conference was an excellent forum for current breaking work. At the same time, the principal speakers paid careful attention to the pedagogical needs of the more junior participants with gentle introductory remarks and surveys.

2 Executive summary

Teichmüller theory has evolved from a sub-field of complex analysis in the 1950-1960s, to a standard tool in low-dimensional topology in the 1970s, to its combinatorial formulation in the 1980s, now with applications across a broad spectrum of fields in mathematics and in physics. Of course, from its very foundations, Teichmüller theory studies geometric structures on surfaces, so its application in low-dimensional geometry and dynamics is of no surprise.

To explain its likewise expected connections with certain aspects of physics, let us recall that the Teichmüller space of a surface is the collection of all conjugacy classes of discrete and faithful representations of its fundamental group into the particular Lie group G=PSL(2,R). This is to be contrasted with the moduli space of all flat connections on a principal G bundle over the surface for an arbitrary Lie group G of central interest in gauge theory, which admits the identical definition simply dropping the provisos of discreteness and faithfulness. This already explains the sense in which gauge theory is a close cousin of Teichmüller theory. It is furthermore natural by analogy to consider the subspace of the moduli space of flat G connections corresponding to discrete and faithful representations, or the so-called higher Teichmüller spaces. Whereas points of Teichmüller space correspond via the Uniformization Theorem to hyperbolic structures on surfaces, it has been a central open problem for several years to see to what intrinsic geometric surface structures might these points of higher Teichmüller space correspond. This important problem has been very recently answered by Marc Burger, Olivier Guichard, Alessandra Iozzi and Anna Wienhard, as reported at the conference by Anna Wienhard: points of these higher Teichmüller spaces correspond to deformation spaces of explicit geometric structures on explicit compact bundles over surfaces.

A rather surprising aspect of this breadth of application is that the decorated Teichmüller spaces provide the most elementary and indeed prototypical examples of cluster varieties, which find expression in representation theory, combinatorics, quantum groups, quantization, and beyond. In this realm at the conference, Andrei Zelevinsky, one of the progenitors of cluster varieties, proposed the challenge of discovering the geometric meaning of certain explicit algebraic structures called g-vectors and F-polynomials which seem to unify the two different systems of cluster coordinates. Volodya Fock furthermore described an explicit construction parallel to geometric quantization for any cluster variety.

The moduli space of Riemann surfaces is the quotient of the Teichmüller space by the mapping class group of the underlying surface, i.e., the group of homotopy classes of orientation-preserving homeomorphisms of the surface. (In unfortunate terminological contrast to the moduli space of flat connections discussed previously, it is Teichmüller space, not Riemann's moduli space, that is akin to moduli spaces in gauge theory.) In fact, there is a mapping class group invariant ideal cell decomposition of decorated Teichmüller space that has provided a purely combinatorial description of Riemann's moduli space in terms of so-called fatgraphs, and at the same time, a new combinatorial groupoid, called the Ptolemy groupoid, which contains the mapping class group as the stabilizer of any object. This cell decomposition is intimately connected with the highly studied string topology, and Nathalie Wahl presented at the conference results extending certain computations in string topology based on fatgraphs. Furthermore, the quantum invariants of three-dimensional manifolds that arise from Chern-Simons gauge theory were interpreted by Jean-Baptiste Meilhan at the conference in terms of an appropriate action of the Ptolemy groupoid. Moreover, a model for open/closed string theory was presented at the conference by Ralph Kaufmann giving generators and relations for topological field theory as well as a specialization to conformal field theory using the ideal cell decomposition.

Despite the apparent utility of this ideal cell decomposition towards this end, the homology of Riemann's moduli space is not yet known explicitly, and this is a fundamental open computation spanning a number of fields in math and physics. However, there is a version of homology stability for these spaces, and the stable homology is understood. Nevertheless, closely related important calculations do not stabilize, for example, the homology of the so-called Torelli subgroup of the mapping class group consisting of those mapping classes that act identically on the homology of the surface. In his presentation at the conference, Benson Farb introduced a new weaker notion of stability for sequences of groups based upon their representation theory and proposed this as a model for a number of phenomena across mathematics including the Torelli groups. He has so far only proved this new version of stability for the symmetric groups, but this already has led to new results about symmetric groups and to new simpler proofs of known results in algebraic number theory. This seems to be a very exciting and broad frontier presented at the conference.

A long-standing question is whether the mapping class groups have Kazhdan's Property T, which avers that any representation of a group sufficiently near the identity in the so-called Fell topology has a fixed vector (roughly, a representation of a group on a Hilbert space with arbitrarily nearly fixed vectors necessarily has a fixed vector). In a remarkable application of his Toeplitz quantization of the moduli spaces of flat G=SU(n) connections for various n, Joergen Ellegaard Andersen reported at the conference his proof that mapping class groups do not have this property. This lecture on the general framework for geometric quantization by Andersen provoked Norbert A'Campo to revisit his earlier consideration of a Topological Quantum Field Theory based on measured foliations, and he appears to have outlined at the conference a new and purely topological construction enjoying all the key properties of Andersen's quantization.The details have still to be checked. At the same time, A'Campo's own presentation at the conference was inspiring and gave a new and elegant method of describing holonomies of singularities as suitably equivariant equivalence relations rather than the more awkward and traditional description as a subgroup of an appropriate mapping class group.

Another long-standing question due to Ehrenpreis is whether "Teichmüller theory disappears virtually", i.e., given a fixed bound, do any two fixed Riemann surface structures of possibly different topological type on specific surfaces admit finite-sheeted unbranched covers of distance less than the fixed bound? One may ask this question for various metrics on Teichmüller space and in the setting of either punctured or unpunctured surfaces. (In effect, the punctured version now allows a controlled branching but only over the now absent punctures.) Vlad Markovic at the conference reported the solution to this question in the affirmative for the Weil-Petersson metric in the punctured and unpunctured cases and his belief that he can also solve this problem for the Teichmüller metric in the unpunctured case. This is already fantastic progress on a motivating and famous problem in the field. As if it were not enough, he also presented as a further application of the same techniques the positive resolution to the surface subgroup problem that the fundamental group of any hyperbolic 3-manifold contains some non-trivial fundamental group of a surface, which is again a long-standing question in 3-manifold theory.

Kirill Krasnov and Jean-Marc Schlenker spoke on interactions with Teichmüller theory with anti-de Sitter geometry.

John Smillie, Ursula Hamenstadt, and Gabriele Mondello spoke on dynamics of billiards and dynamics of flows on spaces of quadratic differentials

Scott Wolpert presented aspects of the background geometry for the very recent breakthrough of Burns-Masur-Wilkinson that the Weil-Petersson geodesic flow on Teichmüller space is ergodic.

3 Scientific content

There were 15 invited speakers, each of whom gave a 55-minute lecture with 5 minutes for questions, and we begin with a very short synopsis of each of these presentations.

Norbert A'Campo from the University of Basel spoke on recent work giving an elegant new formalism for codifying holonomy groups of singularities in terms of equivariant equivalence relations; these methods were illustrated with a number of explicit examples for singularities of plane curves.

Nathalie Wahl of Copenhagen University then spoke on her work generalizing recent results of Costello giving computations of Hochschild homology groups of algebras arising in string topology; indeed, a number of such computations result from her general categorical constructions.

Ursula Hamenstadt of Bonn University discussed her work studying the Teichmüller flow on components of the space of quadratic differentials, which is based on train tracks; deep and basic open questions remain about the topology of these components which we have just begun to probe.

Bill Goldman from the University of Maryland presented his joint work with a number of collaborators on flat Riemannian manifolds, which have been effectively classified; the considerably more subtle classification of manifolds with flat connections was discussed with special attention to the case of dimension three.

Kirill Krasnov from the University of Nottingham spoke on his joint work with Jean-Mark Schlenker relating anti-de Sitter with hyperbolic geometry, in particular showing that a certain renormalized volume based on physical considerations gives a potential for the Weil-Petersson Kaehler structure on Teichmüller space; this leads to new and simpler proofs of results in 2-dimensional geometry relying on tools from 3-dimensional geometry.

Vladimir Markovic of Warwick University discussed his joint work with Jeremy Kahn based on so-called thin pants in 3-manifolds emphasizing the combinatorics and topology and suppressing the necessary analytic estimates; they have used these techniques to great effect and solved a number of famous problems in geometry and dynamics including the surface subgroup problem for 3-manifolds and various versions of the Ehrenpreis Conjecture.

Joergen Ellegaard Andersen of Aarhus University presented his work on the Toeplitz quantization of the moduli spaces of flat connections on a surface as a model of topological quantum field theory; using these tools, he has resolved several famous problems including the asymptotic faithufulness of these quantum representations of mapping class groups as conjectured by Turaev as well as the unexpected application to geometric group resolving in the negative the long-standing question of whether the mapping class groups satisfy Kazhdan's Property T.

Vladimir Fock of Strasbourg University spoke on his current joint work with Sasha Goncharov which generalizes the usual set-up of geometric quantization of suitable symplectic manifolds to arbitrary cluster varieties; rather than the usual line bundles, vector bundles with a projectively flat connection are produced from the cluster structure depending upon a rational parameter derived from the rank and dimension.

John Smillie of Cornell University presented work on renormalization of polygonal billiards, which is related to the Teichmüller flow; elementary considerations in effect predict the appearance of classical continued fractions in the simplest case of square billiards, and other geometries likewise lead to interesting analogous dynamics still in dimension one.

Gabriele Mondello of University of Roma at Sapienza spoke on his work on dynamics on Teichmüller spaces showing how a family of local flows of the general linear group on the cotangent bundle deform to the usual flows on the space of quadratic differentials.

Andrei Zelevinsky of Northeastern University gave an introduction to cluster varieties leading up to the proposal that the two standard sets of variables are specializations of a common ancestry yet to be understood and manifest as the so-called g-vectors and F-polynomials; he provided the challenge to discover the geometric meaning of these ancestors in the special case of cluster varieties arising from surface triangulations.

Anna Wienhard of Princeton University presented her joint work with Olivier Guichard of the last few years which provides beautiful answers to the questions of what are the intrinsic geometric structures on surfaces which correspond to the so-called higher Teichmüller spaces of faithful and discrete representations of surface groups into suitable Lie groups as arise naturally in gauge theory.

Benson Farb from the University of Chicago spoke on his joint work with his current student Tom Church which introduces a new notion of stability for sequences of groups, which they call representation stability, intended to provide a more prevalent phenomenon than the homological stability observed for braid groups, mapping class groups, and symmetric groups; they prove this representation stability for the symmetric groups providing novel proofs of several standard results in number theory and propose a sweeping conjectural framework for applying these new ideas.

Jean-Marc Schlenker from Toulouse University presented his joint work with Francesco Bonsante on dynamics of quasisymmetric homeomorphisms of the circle based on techniques from physics on 3-dimensional anti-de Sitter space; a proof was given of a geometric analogue of Schoen's Conjecture that harmonic quasiconformal extensions uniquely exist.

Scot Wolpert from the University of Maryland presented an overview of the Weil-Petersson geometry of Teichmüller space starting from first principles and leading to explanations and derivations of estimates for the full metric, the covariant derivative of the Levi-Civita connection, and the Riemann curvature tensor; recent applications to the work of Liu-Sun-Yau and Burns-Masur-WIlkinson were also sketched.

There were furthermore shorter presentations (45 minutes each) given by Ralph Kaufmann (Purdue University), Luis Paris (University of Bourgogne), Takuya Sakasai (Tokyo Institute of Technology), Robert Frigerio (University of Padova), Vladimir Vershinin (University of Montpelier), Roland Knevel (University of Luxemburg), Halic Mihai (King Fahd University), Moira Chas (Stonybrook University), David Radnell (American University of Sharjah), Eric Shippers (University of Manitoba), Jean-Baptiste Meilhan (University of Grenoble), and Anna Felikson (Independent University of Moscow).

4 Assessment of the results and forward look

Here is a list of research directions that came out at the conference.

A'Campo's measured foliation treatment of geometric quantization.

Describe the Teichmüller length spectrum of Riemann's moduli space and the associated number fields possibly using TQFT in its Toeplitz and/or A'Campo's new measured foliation incarnation

Explain Zelevinsky's g-vectors and F-polynomials in decorated Teichmüller theory terms.

Identifying the compactification of Riemann's moduli space pertinent to string topology and/or symplectic field theory.

Classify higher-dimensional affine structures.

Realize Farb's far-reaching program for developing the techniques of representation stability.

The Ehrenpreis conjecture for Teichmüller metric.

Extend the ideal cell decomposition to augmented Teichmüller space.

Extend the arc complex topological field theory and/or the string topology to fermions.

Understand quantum invariants arising from perturbation other than at the trivial flat connection.

A complete classification of higher Teichmüller spaces.

Further develop the commonalities between anti-de Sitter and hyperbolic geometry.

Applications of Teichmüller and moduli space techniques to enumerative and predictive problems in both proteomics and genomics.